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A PROPERTY OF SQUARE-FREE INTEGERS

BY

L. MIRSKY, *University of Sheffield.*

[Received 12 July, 1948.]

1. Estermann[1] has shown that if q is any non-zero integer, then there exist infinitely many square-free integers of the form n^2+q. The object of the present note is to establish the following result.

THEOREM. *Let $s \geqslant 1$, and let $q_1, \ldots, q_s$ be any non-zero integers. The set* **E** *of positive integers n such that $n^2+q_1, \ldots n^2+q_s$ are all square-free is either empty or infinite. More precisely, if $E(x)$ denotes the number of integers in* **E** *which do not exceed x, then, when* **E** *is non-empty,* $\liminf_{x \to \infty} E(x)/x > 0$.

I have, in fact, obtained more general and more precise results which I hope to publish elsewhere. However, as some of the proofs are fairly long and involved it seems worth while to see how far a really simple argument will take us.

2. Assume that **E** is non-empty, i.e. there exists a positive integer n_0 such that $n_0^2+q_1, \ldots, n_0^2+q_s$ are all square-free. Write

$$q = \max_{1 \leqslant k \leqslant s} |q_k|.$$

Denote by p_0 the greatest prime factor[2] of $|q_1 \ldots q_s|$, and choose $y > p_0$ such that

$$\sum_{p > y} \frac{1}{p^2} < \frac{1}{6s}. \tag{1}$$

1. T. Estermann, Einige Sätze über quadratfreie Zahlen, *Math. Annalen*, 105 (1931), 654-662.

2. The greatest prime factor of 1 may be conveniently interpreted as 1.

Let
$$c = \prod_{p \leqslant y} p^2,$$
and denote by **D** the set of integers n given by $n = n_0 + mc$ ($m = 0, 1, 2, \ldots$).

For any $1 \leqslant k \leqslant s$ and any prime p, we define
$$F_k(x, p) = \sum_{n_0 + mc \leqslant x}{}' 1;$$
here (and also subsequently) the accent indicates that m is subject to the additional condition
$$(n_0 + mc)^2 + q_k \equiv 0 \pmod{p^2}. \tag{2}$$

Throughout the proof x will be taken as sufficiently large; $\pi(x)$ denotes, as usual, the number of primes not exceeding x.

3. If $p \leqslant y$, then $p^2 | c$ and (2) is clearly insoluble since $n_0^2 + q_k$ is square-free. Hence
$$F_k(x, p) = 0 \ (p \leqslant y). \tag{3}$$

Consider next the case when $p > y$. The congruence (2) is not soluble unless
$$n^2 + q_k \equiv 0 \pmod{p^2} \tag{4}$$
is soluble. But $p > y$ implies that p does not divide q_k, and therefore (4) is either insoluble or else has precisely two incongruent solutions, say n_1 and n_2. In the latter case (2) will be satisfied by precisely those m which satisfy either $n_0 + mc \equiv n_1 \pmod{p^2}$ or $n_0 + mc \equiv n_2 \pmod{p^2}$. Since $(p^2, c) = 1$ we see that the solutions of (2) consist of at most two residue classes (mod p^2). Hence
$$F_k(x, p) \leqslant \frac{2x}{cp^2} + 2, \ (p > y). \tag{5}$$

Furthermore, we have trivially
$$F_k(x, p) = 0, \ (p > x + q). \tag{6}$$

4. Making use of (3), (6), (5), and (1), we have

$$E(x) = \sum_{\substack{n \leqslant x \\ n \,\varepsilon\, \mathrm{E}}} 1 \geqslant \sum_{\substack{n \leqslant x \\ n \,\varepsilon\, \mathrm{E} \\ n \,\varepsilon\, \mathrm{D}}} 1$$

$$\geqslant \sum_{n_0 + mc \leqslant x} 1 - \sum_{1 \leqslant k \leqslant s} \sum_{p} \sum_{n_0 + mc \leqslant x}{}' 1$$

$$\geqslant \frac{x}{2c} - \sum_{1 \leqslant k \leqslant s} \sum_{p} F_k(x, p)$$

$$\geqslant \frac{x}{2c} - \sum_{1 \leqslant k \leqslant s} \sum_{y < p \leqslant x+q} \left\{ \frac{2x}{cp^2} + 2 \right\}$$

$$\geqslant \frac{x}{2c} - \frac{2xs}{c} \sum_{p > y} \frac{1}{p^2} - 2s\pi\,(x+q)$$

$$\geqslant \frac{x}{2c} - \frac{x}{3c} - 2s\pi\,(x+q) \geqslant \frac{x}{7c},$$

and the assertion therefore follows.

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ON A METHOD OF SUMMING DIVERGENT SERIES

BY

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[Received 7 November, 1948.]

1. Let us consider the sum of the infinite series $u_0+u_1+u_2+\ldots$

Let $$s_n = u_0+u_1+\ldots+u_{n-1}. \tag{1}$$

We shall consider those definitions in which the series (1) is said to have the sum t in case $\lim\limits_{r\to\infty} t_r$ exists and has the finite value t, where $t_r = \sum\limits_{k=0}^{\infty} c_{rk}\, s_k$, the c_{rj} being (real or complex) constants.

A sufficient condition that the definition shall be regular is that

(1) $\lim\limits_{r\to\infty} c_{rk} = 0$, for every k;

(2) the series $c_{r0}+c_{r1}+\ldots$ shall converge absolutely and its sum shall approach unity as r approaches infinity;

(3) for every r, the sum of the series

$|c_{r0}|+|c_{r1}|+\ldots$ shall be less than a quantity M independent of r.

Let us take for $\{c_{rk}\}$ the sequence $\{2\, J^2_{r+k}(\omega)\}$. Cooker(1) has shown that the sequence fulfils the conditions (1), (2), (3) mentioned above.

We shall now obtain the Bessel-sum of certain series which represent an arbitrary function in an interval. The series may diverge at certain critical points but the Bessel-sum converges to the value of the function at those points.

2. THEOREM 1. *Let $f(x)$ be absolutely continuous in the closed interval $(-1, 1)$. Then the Legendre series of $f(x)$ is summable $(J, 0)$ to $f(1)$ at $x=1$.*

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The integral

$$S_{n+1}(x) = \int_{-1}^{1} \sum_{n=0}^{n} \left\{ (n+\tfrac{1}{2}) P_n(x) P_n(x') \right\} f(x')\, dx'$$

represents the partial sum of $(n+1)$ terms of Legendre series. The finite sum $\sum_{n=0}^{n} (2n+1) P_n(x) P_n(x')$ was obtained by Christoffel(2) in the form

$$(n+1) \left\{ \frac{P_n(x)\, P_{n+1}(x') - P_{n+1}(x)\, P_n(x')}{x'-x} \right\}.$$

Hence

$$S_{n+1}(x) = \tfrac{1}{2}(n+1) \int_{-1}^{1} \frac{P_n(x)\, P_{n+1}(x') - P_{n+1}(x)\, P_n(x')}{x'-x} f(x')\, dx'.$$

Therefore

$$S_n(1) = \tfrac{1}{2} n \int_{-1}^{1} \frac{P_n(x') - P_{n-1}(x')}{x'-1} f(x')\, dx'$$

$$= \tfrac{1}{2} n \int_{0}^{\pi} \frac{P_n(\cos\theta') - P_{n-1}(\cos\theta')}{\cos\theta' - 1} f(\cos\theta') \sin\theta'\, d\theta'.$$

Now Mehler(3) has shown that

$$P_n(\cos\theta') = \frac{2}{\pi} \int_{0}^{\theta'} \frac{\cos(n+\tfrac{1}{2})\varphi}{\{2(\cos\varphi - \cos\theta')\}^{\frac{1}{2}}}\, d\varphi.$$

Therefore

$$P_n(\cos\theta') - P_{n-1}(\cos\theta') = \frac{2}{\pi} \int_{0}^{\theta'} \frac{\cos(n+\tfrac{1}{2})\varphi - \cos(n-\tfrac{1}{2})\varphi}{\{2(\cos\varphi - \cos\theta')\}^{\frac{1}{2}}}\, d\varphi$$

$$= -\frac{4}{\pi} \int_{0}^{\theta} \frac{\sin n\varphi \sin\tfrac{1}{2}\varphi}{\{2(\cos\varphi - \cos\theta')\}^{\frac{1}{2}}}\, d\phi.$$

Hence

$$S_n(1) = \frac{2}{\pi} \int_{0}^{\theta'} \frac{n \sin n\varphi \sin\tfrac{1}{2}\varphi}{\{2(\cos\varphi - \cos\theta')\}^{\frac{1}{2}}} \frac{f(\cos\theta') \sin\theta'}{1-\cos\theta'}\, d\varphi\, d\theta'.$$

Therefore

$$2\sum \mathcal{J}_n^2(\omega)\, S_n(1) = \frac{1}{\pi} \int_{0}^{\pi} \int_{0}^{\theta'} \frac{\omega \mathcal{J}_1(2\omega \sin\tfrac{1}{2}\varphi) \sin\phi}{\{2(\cos\varphi - \cos\theta)\}^{\frac{1}{2}}}$$

$$\times \frac{f(\cos\theta') \sin\theta'}{1-\cos\theta'}\, d\varphi\, d\theta'$$

which reduces to*

$$\frac{2}{\pi}\int_0^\pi\int_0^{\frac{1}{2}\pi} \omega\, J_1(2\omega \sin \tfrac{1}{2}\theta' \sin \psi)$$

$$\times \sin \tfrac{1}{2}\theta' \sin \psi\, f(\cos \theta') \frac{\sin \theta'}{1-\cos \theta'}\, d\psi\, d\theta'$$

$$= \tfrac{1}{2}\int_0^\pi \omega^2 \sin^2 \tfrac{1}{2}\theta'\ {}_1F_2\left(\tfrac{3}{2}; 2, 2; -\omega^2 \sin^2 \tfrac{1}{2}\theta'\right)$$

$$\times \frac{f(\cos\theta') \sin \theta'}{1-\cos \theta'}\, d\theta'.$$

Let $f(\cos \theta') = 1$ for all values of θ'. We have

$$1 = \frac{2}{\pi}\int_0^\pi\int_0^{\theta'} \frac{n \sin n\phi \sin \frac{1}{2}\phi}{\left\{2(\cos\phi - \cos\theta')\right\}^{\frac{1}{2}}} \frac{\sin \theta'}{1-\cos \theta'}\, d\varphi\, d\theta'.$$

Hence

$$2\Sigma J_n^2(\omega) f(1-0) = \frac{4}{\pi}\int_0^\pi\int_0^{\theta'} \sum \frac{n \sin n\phi \sin \frac{1}{2}\theta}{\left\{2(\cos\phi - \cos\theta')\right\}^{\frac{1}{2}}}$$

$$\times \frac{J_n^2(\omega) f(1-0) \sin\theta'}{1-\cos\theta}\, d\varphi\, d\theta'.$$

We therefore have after slight reductions,

$$2\Sigma J_n^2(\omega)\,(S_n - f(1-0))$$

$$= \tfrac{1}{2}\int_0^\pi \omega^2 \sin^2 \tfrac{1}{2}\theta'\ {}_1F_2\left(\tfrac{3}{2}; 2,2; -\omega^2 \sin^2 \tfrac{1}{2}\theta'\right)$$

$$\times \Big(f(\cos\theta') - f(1-0)\Big) \frac{\sin\theta'\, d\theta'}{1-\cos\theta'}.$$

Hence the theorem is true, provided that

$$\lim_{\omega\to\infty} \int_0^\pi \omega^2 \sin\theta'\ {}_1F_2\left(\tfrac{3}{2}; 2,2; -\omega^2 \sin^2 \tfrac{1}{2}\theta'\right)$$

$$\times \left(f(\cos\theta') - f(1-0)\right) d\theta' \to 0.$$

Let

$$I = \int_0^\pi \omega^2 \sin\theta'\ {}_1F_2\left(\tfrac{3}{2}; 2,2; -\omega^2 \sin^2 \tfrac{1}{2}\theta'\right)\Big(f(\cos\theta') - f(1-0)\Big)\, d\theta'.$$

* The inversion of the order of integration and summation can be justified by the Lebesgue convergence theorem. Later on we make use of the identity $\cos(x \sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \ldots$ whence we obtain easily

$$xJ_1(2x\sin\theta)\cos\theta = 2\left\{J_1^2(x)\sin 2\theta + 2J_2^2(x)\sin 4\theta + \ldots\right\}.$$

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Let $\omega^2 \sin^2 \frac{1}{2}\theta' = z.$

Then

$$I = \frac{1}{2}\int_0^{\omega^2} {}_1F_2\left(\tfrac{3}{2};\, 2{,}2\,;-z\right)\Big(f\big(1-\tfrac{2z}{\omega^2}\big)-f(1-0)\Big)\,dz.$$

Since

$$J_0^2(z) = {}_1F_2\left(\tfrac{1}{2};\, 1\,,1\,;-z^2\right), \text{ we have}$$

$$\frac{d}{dz} J_0^2(\sqrt{z}) = -\tfrac{1}{2}\,{}_1F_2\left(\tfrac{3}{2};\, 2{,}2\,;-z\right).$$

Hence

$$I = -\int_0^{\omega^2} \frac{d}{dz}\left\{J_0^2(\sqrt{z})\right\}\left[f\big(1-\tfrac{2z}{\omega^2}\big)-f(1-0)\right]dz.$$

Integrating by parts,

$$I = \Big[J_0^2(\sqrt{z})\,\big[f\big(1-\tfrac{2z}{\omega^2}\big)-f(1-0)\big]\Big]_{\omega^2}^{0} + \int_0^{\omega^2} J_0^2(\sqrt{z})\,\frac{d}{dz}f\Big(1-\frac{2z}{\omega^2}\Big)dz.$$

The second part under the sign of integration can be written as

$$-\int_{-1}^{1} J_0^2\Big(\sqrt{\frac{1-u}{2}}\,\omega\Big)\,\frac{d}{du}f(u)\,du.$$

The first part vanishes since $f(x)$ is continuous at $x=1$ and $J_0(\omega) \to 0$ as $\omega \to \infty$.

The second part can be written as $\int_{-1}^{1-\delta} + \int_{1-\delta}^{1} = I_1 + I_2.$

Now $I_1 \to 0$ as $\omega \to \infty$ for a given δ and $I_2 \to 0$ as $\delta \to 0$ since $f'(x)$ exists p.p. in $(-1, 1)$ and is summable in $(-1, 1)$ under the hypothesis.

THEOREM 2. *Let $f(x)$ be a function which has a continuous differential coefficient with limited total fluctuation for all values of x in the closed interval $(0, \pi)$. Then the infinite series of Bessel functions of order zero, corresponding to the expansion of $f(x)$ is summable $(J, \frac{1}{2})$ to $f(0)$ at the origin.*

Schlömilch[4] has proved that $f(x)$ may be expanded in the series

$$f(x) = a_0 + a_1 \mathcal{J}_0(x) + a_2 \mathcal{J}_0(2x) + \ldots,$$

where $$a_0 = f(0) + \frac{1}{\pi}\int_0^{\pi} u \int_0^{\frac{1}{2}\pi} f'(u \sin\theta)\, d\theta\, du$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} u \cos nu \int_0^{\frac{1}{2}\pi} f'(u \sin\theta)\, d\theta\, du, \ (n > 0).$$

The only critical points where the infinite series may diverge are zero and $\pm\pi$. We shall show that $f(x)$ is summable $(\mathcal{J}, \frac{1}{2})$ to $f(0)$ at $x = 0$.

Let
$$S_{n+1}(0) = a_0 + a_1 + \ldots + a_{n+1}$$
$$= f(0) + \frac{1}{\pi}\int_0^{\pi} \frac{u \sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u} du \int_0^{\frac{1}{2}\pi} f'(u\sin\theta)\, d\theta.$$

Hence

$$2\Sigma \mathcal{J}^2_{n+\frac{1}{2}}(\omega)\, S_{n+1}(0) = \left(1 - \int_{\omega}^{\infty} \mathcal{J}^2_{\frac{1}{2}}(t)\frac{dt}{t} + \mathcal{J}^2_{\frac{1}{2}}(\omega)\right) f(0)$$
$$+ \frac{1}{\pi}\int_0^{\pi} \frac{2u\,\Sigma\mathcal{J}^2_{n+\frac{1}{2}}(\omega)\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u} du \int_0^{\frac{1}{2}\pi} f'(u\sin\theta)\, d\theta.$$
$$= \left(1 - \int_{\omega}^{\infty} \mathcal{J}^2_{\frac{1}{2}}(t)\frac{dt}{t} + \mathcal{J}^2_{\frac{1}{2}}(\omega)\right) f(0)$$
$$+ \frac{1}{\pi}\int_0^{\pi} \frac{u H_0(2\omega\sin\frac{1}{2}u)}{\sin\frac{1}{2}u} du \int_0^{\frac{1}{2}\pi} f'(u\sin\theta)\, d\theta;$$

the inversion being justified by the Lebesgue convergence theorem. Also we notice that $\int_{\omega}^{\infty} \mathcal{J}^2_{\frac{1}{2}}(t)\frac{dt}{t}$ and $\mathcal{J}^2_{\frac{1}{2}}(\omega) \to 0$ as $\omega \to \infty$ and $|H_0(2\omega\sin\frac{1}{2}u)| \leqslant 1$.

Let
$$\psi(u) = \int_0^{\frac{1}{2}\pi} f'(u\sin\theta)\, d\theta.$$

Then $\psi(u)$ is continuous.(5) Writing

$\int_0^{\pi} \frac{u H_0(2\omega\sin\frac{1}{2}u)}{\sin\frac{1}{2}u}\psi(u)\, du = \int_0^{\eta} + \int_{\eta}^{\pi} = I_1 + I_2$, we find that $|I_1| < A\eta$ and $|I_2| \to 0$ as $\omega \to \infty$, since $H_0(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \times \sin\left(x - \frac{\pi}{4}\right) + o\left(\frac{1}{x}\right)$. By choosing η sufficiently small, we find that $|I_1| < \varepsilon$ and the theorem is proved.

4. THEOREM 3. *Let $f(x)$ be a bounded function integrable L in $(0, 2\pi)$. Let its Fourier series be*

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Then its conjugate series

$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx)$ *is summable* $(J, 0)$ *to*

$$f_1(x) = -\frac{1}{2\pi}\int_0^{\pi} [f(x+u) - f(x-u)] \cot \tfrac{1}{2} u \, du,$$

provided that

$$\operatorname*{Lt}_{\omega\to\infty} \int_0^{\delta} \frac{J_0(2\omega \sin \frac{1}{2}u) \cos \frac{1}{2}u}{\sin \frac{1}{2}u} [f(x+u) - f(x-u)] \, du \to 0,$$
$$(0 < \delta < \pi).$$

PROOF. Let

$$S_n = \sum_{n=1}^{n} (a_n \sin nx - b_n \cos nx) = \frac{1}{\pi}\sum_{n=1}^{n}\int_0^{2\pi} \sin n(x-t) f(t)\, dt.$$

Since

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t) \cos nt \, dt, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(t) \sin nt \, dt,$$

therefore

$$S_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin (n+1)/2\ (x-t) \sin n/2\ (x-t)}{\sin (x-t)/2} f(t)\, dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos \frac{1}{2}(x-t) - \cos (n+\frac{1}{2})(x-t)}{\sin (x-t)/2} f(t)\, dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos (n+\frac{1}{2})(t-x) - \cos \frac{1}{2}(t-x)}{\sin (t-x)/2} f(t)\, dt.$$

Let $t = u + x$, then

$$S_n = \frac{1}{2\pi}\int_{-x}^{2\pi-x} \frac{\cos (n+\frac{1}{2}) u - \cos \frac{1}{2}u}{\sin \frac{1}{2}u} f(x+u)\, du$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos (n+\frac{1}{2}) u - \cos \frac{1}{2}u}{\sin \frac{1}{2}u} f(x+u)\, du$$
$$= \frac{1}{2\pi}\int_0^{\pi} \frac{\cos (n+\frac{1}{2}) u - \cos \frac{1}{2}u}{\sin \frac{1}{2}u} [f(x+u) - f(x-u)]\, du.$$

Therefore if $S_1 = f_1(x)$, we get*

$$S_n - S_1 = \frac{1}{2\pi}\int_0^\pi \frac{\cos(n+\frac{1}{2})u}{\sin\frac{1}{2}u}[f(x+u)-f(x-u)]\,du,$$

and

$$S_{n-1} - S_1 = \frac{1}{2\pi}\int_0^\pi \frac{\cos(n-\frac{1}{2})u}{\sin\frac{1}{2}u}[f(x+u)-f(x-u)]\,du.$$

Therefore

$$\frac{S_n+S_{n-1}}{2} - S_1 = \frac{1}{\pi}\int_0^\pi \frac{\cos nu\cos\frac{1}{2}u}{\sin\frac{1}{2}u}[f(x+u)-f(x-u)]\,du.$$

Hence

$$2\sum_{n=0}^{\infty} \mathcal{J}_n^2(\omega)\left(\frac{S_n+S_{n-1}}{2} - S_1\right)$$

$$= \frac{2}{\pi}\int_0^\pi \frac{\Sigma \mathcal{J}_n^2(\omega)\cos nu\cos\frac{1}{2}u}{\sin\frac{1}{2}u}[f(x+u)-f(x-u]\,du$$

$$= \int_0^\pi \frac{(\mathcal{J}_0^2(\omega)+\mathcal{J}_0(2\omega\sin\frac{1}{2}u))\cos\frac{1}{2}u}{\sin\frac{1}{2}u}[f(x+u)-f(x-u)]\,du;$$

again justifying the inversion of the order of summation and integration by the Lebesgue convergence theorem.

Hence the conjugate series is summable $(\mathcal{J}, 0)$ to $f_1(x)$ provided that

$$\underset{\omega\to\infty}{\mathrm{Lt}}\int_0^\pi \frac{\mathcal{J}_0(2\omega\sin\frac{1}{2}u)\cos\frac{1}{2}u}{\sin\frac{1}{2}u}[f(x+u)-f(x-u)]\,du \to 0.$$

Since

$$\mathcal{J}_0(2\omega\sin\tfrac{1}{2}u) \sim \frac{1}{\sqrt{\omega\sin\frac{1}{2}u}},$$

$$\int_\delta^\pi \frac{\mathcal{J}_0(2\omega\sin\frac{1}{2}u)\cos\frac{1}{2}u}{\sin\frac{1}{2}u}[f(x+u)-f(x-u)]\,du \to 0 \text{ as } \omega\to\infty.$$

Hence the condition can be replaced by

$$\underset{\omega\to\infty}{\mathrm{Lt}}\int_0^\delta \frac{\mathcal{J}_0(2\omega\sin\frac{1}{2}u)\cos\frac{1}{2}u}{\sin\frac{1}{2}u}[f(x+u)-f(x-u)]\,du \to 0,$$

$$(0<\delta<\pi).$$

* It is assumed that $f_1(x)$ exists.

COR. 1. Suppose $f'(x)$ exists at x. Then

$$\left|\frac{f(x+u)-f(x-u)}{u}\right|$$

$$=\left|\frac{f(x+u)-f(x)}{u}+\frac{f(x-u-f(x)}{-u}\right| \leqslant k, \; (|u| \leqslant \eta).$$

Then as befcre we can prove, by taking first η small enough and then keeping η fixed and making ω tend to infinity, that the integral tends to zero.

COR. 2. Suppos $f(x)$ satisfies Lipschitz's condition of order α, $(0 < \alpha \leqslant 1)$.

Then

$$|f(x+u)-f(x)| < k\,|u|^{\alpha}, \text{ when } |u| \leqslant \eta,$$

and

$$\int_0^{\delta}[f(x+u)-f(x-u)]\,\frac{J_0(2\omega \sin \tfrac{1}{2}u)\,\cos \tfrac{1}{2}u}{\sin \tfrac{1}{2}u}\,du$$

$$=\int_0^{\eta}+\int_{\eta}^{\delta}=I_1+I_2.$$

Hence

$$|I_1| \leqslant \int_0^{\eta}\frac{|f(x+u)-f(x-u)|}{\sin \tfrac{1}{2}u}\,du \leqslant A\int_0^{\eta}u^{\alpha-1}\,du=\frac{A}{\alpha}\,\eta^{\alpha},$$

where A is a constant. The integral tends to zero as $\eta \to 0$.

The second integral

$$|I_2| \leqslant \int_{\eta}^{\delta}|f(x+u)-f(x-u)|\,\frac{|J_0(2\omega \sin \tfrac{1}{2}u)|}{\sin \tfrac{1}{2}u}\cos \tfrac{1}{2}u\,du \to 0$$

$$\text{as } \omega \to \infty,$$

since $f(x)$ is bounded in the interval.

COR. 3. Let

$$\frac{f(x+u)-f(x-u)}{u}$$

be bounded and integrable, then

$$\int_0^{\delta}[f(x+u)-f(x-u)]\,\frac{J_0(2\omega \sin \tfrac{1}{2}u)\,\cos \tfrac{1}{2}u}{\sin \tfrac{1}{2}u}\,du$$

$$=\int_0^{\eta}+\int_{\eta}^{\delta}=I_1+I_2,$$

since

$$\left|\frac{f(x+u)-f(x-u)}{u}\right| < k \text{ and } \frac{u}{\sin \frac{1}{2}u}$$

is bounded in $(0, \pi)$, we can by taking η sufficiently small prove that

$$|I_1| < \varepsilon \text{ and } |I_2| \to 0 \text{ as } \omega \to \infty.$$

COR. 4. Let

$$\Phi(t) = \int_0^t \left|\frac{f(x+u)-f(x)}{u}\right| du = o\ (1).$$

Then $|\Phi(t)| < t$, when $|t| \leqslant \eta$.

Let x be a point where the condition holds good. Then

$$\int_0^t \left|\frac{f(x+u)-f(x)-f(x-u)+f(x)}{u}\right| du$$

$$\leqslant \int_0^t \left|\frac{f(x+u)-f(x)}{u}\right| du + \int_0^t \left|\frac{f(x-u)-f(x)}{u}\right| du = o\ (1).$$

Hence

$$\int_0^\delta [f(x+u)-f(x-u)] \frac{J_0(2\omega \sin \frac{1}{2}u) \cos \frac{1}{2}u}{\sin \frac{1}{2}u} du$$

$$= \int_0^\eta + \int_\eta^\delta = I_1 + I_2.$$

Then $|I_1| \leqslant A\varepsilon$, where A is a constant and $|I_2| \to 0$ as $\omega \to \infty$. When η is taken sufficiently small, (say less than ε/A), $|I_1| < \varepsilon$, and the theorem is proved.

5. THEOREM 4. *Let $f(x)$ be a function integrable L in $(0, 2\pi)$. Let $\sum_{n=1}^{\infty} a_n \sin nx$ be the Fourier sine series of $f(x)$. Then it is summable $(J, \frac{1}{2})$ to $f(x)$ provided that**

$$\operatorname*{Lt}_{\omega\to\infty} \int_0^\delta [f(x+u)+f(x-u)-2f(x)] \frac{H_0(2\omega \sin \frac{1}{2}u)}{\sin \frac{1}{2}u} du \to 0,$$

$$(0 < \delta < \pi).$$

* We can replace $2f(x)$ by $f(x+0)+f(x-0)$.

We can easily prove that

$$S_n = \frac{1}{4\pi}\int_0^{2\pi} [f(x+u)+f(x-u)]\frac{\sin\,(n+\frac{1}{2})\,u}{\sin\,\frac{1}{2}u}\,du.$$

Now

$$1 = \frac{1}{\pi}\int_0^{\pi}\frac{\sin\,(n+\frac{1}{2})\,u}{\sin\,\frac{1}{2}u}\,du.$$

Therefore

$$S_n - S = \frac{1}{2\pi}\int_0^{\pi}\left\{f(x+u)+f(x-u)-2s\right\}\frac{\sin\,(n+\frac{1}{2})\,u}{\sin\,\frac{1}{2}u}\,du.$$

Hence

$$2\sum_{n=0}^{\infty} J^2_{n+\frac{1}{2}}(\omega)\,(S_n - S)$$
$$= \frac{1}{2\pi}\int_0^{\pi}\left\{f(x+u)+f(x-u)-2S\right\}\frac{H_0\,(2\omega\sin\frac{1}{2}u)}{\sin\,\frac{1}{2}u}\,du,$$

where H_0 is Struve's function of order zero. The change in the order of integration and summation can be easily justified by Lebsgue's convergence theorem.

The asymptotic expansion of

$$H_0(x) \text{ is } \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\sin\left(x-\frac{\pi}{4}\right)+O\left(\frac{1}{x}\right).$$

Hence

$$\left|\int_\delta^{\pi}\frac{H_0(2\omega\sin\frac{1}{2}u)}{\sin\,\frac{1}{2}u}\left[f(x+u)+f(x-u)-2S\right]du\right|$$
$$= (\pi\omega)^{-\frac{1}{2}}\int_\delta^{\pi}(\sin\tfrac{1}{2}u)^{-\frac{3}{2}}\left|f(x+u)+f(x-u)-2S\right|du+O\left(\frac{1}{\omega}\right),$$

and $|f(x+u)+f(x-u)-2S|/(\sin\frac{1}{2}u)^p$, where p is any real number, is integrable over (δ, π), where $0<\delta<\pi$, so that the integral tends to zero as $\omega\to\infty$. Thus the theorem is proved by subtraction.

COR. 1. Suppose $f(x)$ is bounded and possesses right hand and left hand derivatives at x.

Let $2s = f(x+0)+f(x-0)$. Since $f(x\pm 0)$ exist and

$$\left|\frac{f(x\pm u)-f(x\pm 0)}{\pm u}\right|\leqslant k, \text{ when } |u|\leqslant\eta,$$

then if we put

$$\Phi(u) = f(x+u) + f(x-u) - f(x+0) - f(x-0),$$

we get
$$\left|\frac{\Phi(u)}{u}\right| \leqslant 2k, \text{ when } |u| \leqslant \eta.$$

Hence

$$\int_0^\delta \frac{H_0\,(2\omega \sin \frac{1}{2}u)}{\sin \frac{1}{2}u}\,\Phi(u)\,du = \int_0^\eta + \int_\eta^\delta = I_1 + I_2,$$

$$|I_1| \leqslant A\int_0^\eta \left|\frac{\Phi(u)}{u}\right| du \leqslant 2Ak\eta, \text{ by taking } \eta < \frac{\varepsilon}{2Ak},$$

$$|I_1| < \varepsilon \text{ and } |I_2| \to 0 \text{ as } \omega \to \infty.$$

The theorem holds good in the following cases.

COR. 2. Let $f(x)$ be bounded and differentiable at x.

COR. 3. Let $f(x)$ satisfy Lipschitz's condition of order α, $(0 < \alpha \leqslant 1)$.

COR. 4. Let $f(x)$ satisfy the relation

$$\int_0^t \left|\frac{f(x+u)-f(x)}{u}\right| du = o\,(1) \text{ at a point } x.$$

COR. 5. Let $\dfrac{\Phi(u)}{u}$ be bounded in the closed interval $(0, \pi)$ and integrable, where

$$\Phi(u) = f(x+u) + f(x-u) - 2f(x).$$

The proofs are on the same lines as before.

COR. 6. Let $\psi(t) = \int_0^t \Phi(u)\,du$. Suppose $\psi(t) \to 0$ as $t \to 0$, and $\psi'(t)$ exists in the closed interval $(0, \delta)$. Then we assert that the sum converges to $f(x)$, where $f(x)$ is continuous at x.

Here
$$\phi(t) = \frac{d}{dt}(t\psi(t)) = \psi(t) + t\psi'(t).$$

Since $\psi'(t)$ exists everywhere in $(0, \delta)$, $|\psi'(t)| < k_1$ and $|\psi(t)/t| < k_2$, when $|t| \leqslant \eta$. Let $k = \max\,(k_1, k_2)$. Then

$$\int_0^\delta \frac{H_0\,(2\omega \sin u/2)}{\sin u/2}\,\Phi(u)\,du = \int_0^\delta \frac{H_0\,(2\omega \sin u/2)}{\sin u/2}\,(\psi(u) + u\psi'(u))\,du$$

$$= \int_0^\eta + \int_\eta^\delta = I_1 + I_2.$$

We prove easily that $|I_1| < \varepsilon$, when η is chosen sufficiently small and $|I_2| \to 0$ as $\omega \to \infty$.

6. THEOREM 5. *Let the Fourier series of $f(x)$ be* $a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$. *The derived series* $\Sigma m\,(b_m \cos mx - a_m \sin mx)$ *is summable* $(J, \frac{1}{2})$ *to* $f'(x)$, *provided that*

$$\underset{\omega\to\infty}{\mathrm{Lt}} \int_0^{\pi} \frac{H_0(2\omega \sin \frac{1}{2} u)}{\sin \frac{1}{2} u} [f'(x+u) + f'(x-u) - 2f'(x)]\, du \to 0.$$

This condition can be easily replaced by

$$\underset{\omega\to\infty}{\mathrm{Lt}} \int_0^{\delta} \frac{H_0(2\omega \sin \frac{1}{2} u)}{\sin \frac{1}{2} u} [f'(x+u) + f'(x-u) - 2f'(x)]\, du \to 0.$$

We assume that $f(x)$ and $f'(x)$ are integrable L in $(0, 2\pi)$ and continuous. Here

$$S_n = \frac{1}{\pi} \sum_{m=1}^{n} \int_0^{2\pi} \frac{d}{dx} \left\{ \cos m\,(t-x) \right\} f(t)\, dt$$

$$= \frac{1}{\pi} \frac{d}{dx} \sum_{m=1}^{n} \int_0^{2\pi} \cos m\,(t-x) f(t)\, dt$$

$$= \frac{1}{\pi} \frac{d}{dx} \int_0^{2\pi} \frac{\cos (n+1)/2\,(t-x) \sin n/2\,(t-x)}{\sin (t-x)/2} f(t)\, dt$$

$$= \frac{1}{2\pi} \frac{d}{dx} \int_0^{2\pi} \frac{\sin (n+\frac{1}{2})(t-x) - \sin \frac{1}{2}(t-x)}{\sin (t-x)/2} f(t)\, dt$$

$$= \frac{1}{2\pi} \frac{d}{dx} \int_{-x}^{2\pi-x} \frac{\sin (n+\frac{1}{2}) u - \sin \frac{1}{2} u}{\sin \frac{1}{2} u} [f(x+u)]\, du$$

$$= \frac{1}{2\pi} \frac{d}{dx} \int_0^{\pi} \frac{\sin (n+\frac{1}{2}) u - \sin \frac{1}{2} u}{\sin \frac{1}{2} u} [f(x+u) + f(x-u)]\, du$$

$$= \frac{1}{2\pi} \int_0^{\pi} \frac{\sin (n+\frac{1}{2}) u - \sin \frac{1}{2} u}{\sin \frac{1}{2} u} [f'(x+u) + f'(x-u)]\, du$$

$$= \frac{1}{2\pi} \int_0^{\pi} \frac{\sin (n+\frac{1}{2}) u}{\sin \frac{1}{2} u} [f'(x+u) + f'(x-u)]\, du.$$

Also

$$1 = \frac{1}{\pi} \int_0^{\pi} \frac{\sin (n+\frac{1}{2}) u}{\sin \frac{1}{2} u}\, du.$$

Therefore

$$S_n - S = \frac{1}{2\pi}\int_0^{\pi} \frac{\sin (n+\frac{1}{2})\, u}{\sin \frac{1}{2} u} [\, f'(x+u)+f'(x-u)-2S)]\, du.$$

$$2\,\Sigma\, J^2_{n+\frac{1}{2}}(\omega)\,(S_n - S^*) = \frac{1}{2\pi}\int_0^{\pi} \frac{H_0(2\omega \sin \frac{1}{2} u)}{\sin \frac{1}{2} u} [\, f'(x+u) + f'(x-u) - 2S]\, du.$$

Hence the series is summable $(J, \frac{1}{2})$ to S, provided that

$$\operatorname*{Lt}_{\omega\to\infty} \int_0^{\pi} \frac{H_0(2\omega \sin \frac{1}{2} u)}{\sin \frac{1}{2} u} [\, f'(x+u)+f'(x-u)-2S]\, du \to 0.$$

We can now replace S by $f'(x)$.

COR. 1. Let $\Phi(u) = f'(x+u)+f'(x-u)-2S$. If $\frac{\Phi(u)}{u}$ is integrable in $(0, \delta)$ then $S_n \to S$.

COR. 2. If $|f'(x+u)-f'(x)| = O(u^{\alpha})$, $(0 < \alpha \leqslant 1)$, then $S_n \to f'(x)$.

COR. 3. $\int_0^{\delta} \left| \frac{f'(x+u)-f'(x)}{u} \right| du = O(1)$, then $S_n \to f'(x)$.

The proofs are on the same lines as before.

I am indebted to the referee for many helpful suggestions.

* The change in the order of summation and integration being justifiable as before.

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CLASSIFICATION OF MULTIPLICATIVE FUNCTIONS OF TWO ARGUMENTS BASED ON THE IDENTICAL EQUATION*

BY

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In an earlier paper [1, 522],† I have established the identity

$$F(M, N) = F(M, 1)\, F(1, N).C(M, N),$$

where $F(M, N)$ is any general multiplicative function of the two arguments M, N and $C(M, N)$ is a cardinal function determined by $F(M, N)$ and called its cardinal component (or briefly the c-component). If $F(M, N)$ is an ordinal or modular function, it is shown [1, 523-25] that the c-component is also *ordinal*. Recently, I have studied [2] a classification of ordinal (and a parallel classification of modular) functions by associating each ordinal (modular) function with an index r, called the precise ordinal (modular) index, where r is the highest number with the property that $F(M, N) = 0$ if $N^r > M$, [$F(M, N)$ is unaltered in value when N is increased by any multiple of M_r, the greatest integer whose rth power divides M]. Further it is proved [2, 96] that if $F(M, N)$ has the precise ordinal modular) index r, the c-component has also *the precise ordinal index* r. A generalisation of principal functions by means of an index was also made there [2, 85]. The generalised principal functions are called quasi-principal functions and have the property $P_r(M, N) = 0$ if $M \neq N^r$; r is called the index and M, the

* Most of the results of this paper were given before, in a thesis, approved for the M. Sc. Degree of the University of Madras and I am indebted to Dr. R. Vaidyanathaswamy for his valuable guidance.

† The numbers in the brackets refer to the references given at the end.

major argument of the quasi-principal function $P_r(M, N)$. The c-component of a quasi-principal function is easily shown to be identical with the function itself [2, 86]. The concept of quasi-principal functions suggested an extension of functions of Ramanujan's class by introducing an index r; thus all modular functions whose c-components are quasi-principal functions with index r, are called functions of Ramanujan's class with the index r.

It is the object of the present note to show how the concept of an index associated with the ordinal, modular and quasi-principal functions and the nature of their c-components lead us to the following

2. *Classifica ion of all Multiplicative Functions of two arguments.*

CLASS I consists of all multiplicative functions $F(M, N)$ whose c-components are ordinal with M as the major argument.

CLASS II consists of all multiplicative functions $F(M, N)$ whose c-components are quasi-principal functions with the major argument M.

CLASS III consists of all multiplicative functions $F(M, N)$ whose c-components are principal functions.

CLASS IV consists of all multiplicative functions $F(M, N)$ whose c-components are non-ordinal functions.

It will be convenient if we refer to the above classes as c_1, c_2, c_3, and c_4 respectively.

All ordinal and modular functions belong to c_1; while all quasi-principal functions and functions of Ramanujan's class belong to c_2. All principal functions are included in c_3. The special interest of c_3 lies in the fact that Ramanujan's function $c_M(N)$ belongs to it [1, 527]. It is clear that $c_1 \supset c_2 \supset c_3$. The classes c_1, c_2 can be sub-divided into c_{1r}, c_{2r}, according to the index. It

follows easily that $c_4 = c_{10}$, [2, 85] and $c_3 = c_{21}$. c_1 and c_2 are respectively the union of c_{1r}'s and c_{2r}'s ($r = 1, 2, \ldots, \infty$) and c_4 is the complementary class of c_1. We may note that $c_{1\infty} = c_{2\infty} = c_{20}$, for $c_{1\infty} \supset c_{2\infty}$ and the only ordinal functions with the major argument M and the index ∞ are of the form $\psi(M)E_0(N)$ and these can be cardinal only if $\psi(M) = E_0(M)$, [2, 84]; in the case of c_{20}, it follows that that $c(M, N) = E_0(M)\, c_1(N) = E_0(M)\, E_0(N) = E_0(M, N)$ because $c(M, N) = 0$ if $M \neq N^0$, i.e. if $M \neq 1$.

We now derive some important properties of the above classes of functions. It is enough if we consider c_1 and c_2, since we have already seen that $c_3 = c_{21}$ and $c_4 = c_{10}$.

THEOREM 1. *Any function of* c_{1r} *can be made into an ordinal (a modular) function with the index* r *by composing it with a suitable function of the minor argument of its c-component.*

For, if $F(M, N)$ belongs to c_{1r} and $c_r(M, N)$ is its c-component, $F(M, N) = F(M, 1)\, F(1, N).c_r(M, N)$. Composing both sides with $F^{-1}(1, N)$, a function of the minor argument of $c_r(M, N)$

$$F^{-1}(1, N).F(M, N)$$
$$= F^{-1}(1, N).F(M, 1)\, F(1, N).c_r(M, N)$$

or* $$E_0(M)F^{-1}(1, N).F(M, N)$$
$$= E_0(M)F^{-1}(1, N).F(M, 1)\, F(1, N).c_r(M, N)$$
$$= \{E_0(M).F(M, 1)\}\, \{F^{-1}(1, N).F(1, N)\}.c_r(M\, N)$$
$$= F(M, 1)\, E_0(N).c_r(M, N).$$

Now $F(M, 1)E_0(N)$ and $c_r(M, N)$ are ordinal functions with the indices ∞, r respectively. Therefore their composite is an ordinal function with the index r [2, 95 Th. 8.2]. Hence the first part of the theorem follows.

On the other hand if we compose $F(M, N)$ with $\{F^{-1}(1, N).E(N)\}$, another function of the minor argument of its c-component,

* Vide definition [1, 519 (1.3.1)].

$$F(M, N).\{F^{-1}(1, N).E(N)\}$$
$$= F(M, N).E_0(M, \{F^{-1}(1, N).E(N)\}$$
$$= F(M, N).E_0(M)\ F^{-1}(1, N).E_0(M)\ E(N)$$
$$= \{F(M, 1)\ E_0(N).c_r(M, N)\}.E_0(M)E(N)$$

= the integral with respect to the minor argument of the ordinal function,

$\{F(M, 1)\ E_0(N).c_r(M, N)\}$ which (we already saw) has the index r. Hence $F(M,N).\{F^{-1}(1, N).E(N)\}$ is a modular function with the index r [2, 91] and the second part of the theorem is established.

THEOREM 2. *Any function of c_{2r} can be transformed into a quasi-principal function with the major argument M and the index r, (or a function of Ramanujan's class with the index r) by composing it with a suitable function of M, N.*

For, if $F(M, N)$ belongs to c_{2r} and $c_r(M, N)$ is its c-component,

$F(M,N) = F(M, 1)\ F(1, N).c_r\ (M, N)$. Composing both sides with $F^{-1}(M, 1)\ F^{-1}(1, N)$,

$$F^{-1}(M, 1)\ F^{-1}(1, N).F(M, N)$$
$$= F^{-1}(M, 1)\ F^{-1}(1, N).F(M, 1)\ F(1, N).c_r(M, N)$$
$$= \{F^{-1}(M, 1).F(M, 1)\}\ \{F^{-1}(1, N)\ F(1, N)\}.c_r(M, N)$$
$$= E_0(M)\ E_0(N).c_r(M, N)$$
$$= E_0(M, N).c_r(M, N)$$

$= c_r(M, N)$, a quasi-principal function of index r with the major argument M.

On the other hand, suppose we compose both sides with $E_0(M)\ \psi(N)$ where $\psi(N) = F^{-1}(1, N).E(N)$.

Let $f(M,N) = F(M,N).E_0(M)\psi(N)$. Then, $f(M, 1) = F(M, 1).E_0(M) = F(M, 1)$ and $f(1, N) = F(1, N).\psi(N) = F(1,N).F^{-1}(1, N).E(N) = E_0(N).E(N) = E(N)$.

Now

$$f(M, N) = F(M, N).E_0(M)\psi(N)$$
$$= E_0(M)\psi(N).F(M, 1)\ F(1, N).c_r(M, N)$$

$$= \{ E_0(M).F(M,1) \} \{ \psi(N).F(\cdot, N) \}.c_r(M,N)$$
$$= F(M,1) \{ F^{-1}(1,N).E(N).F(1,N) \}.c_r(M,N)$$
$$= F(M,1)\, E(N).c_r(M,N)$$
$$= F(M,1)\, E_0(N).c_r(M,N).E_0(M)\, E(N)$$

= the integral with respect to the minor argument of the ordinal function $\{ F(M,1)\, E_0(N).c_r(M,N) \}$ with the index r. Hence $f(M,N)$ is a modular function with the index r.

Further $f(M,N) = F(M,1)\, E(N).c_r(M,N)$
$= f(M,1)\, f(1,N).c_r(M,N)$

showing that the c-component of $f(M,N)$ is the quasi-principal function $c_r(M,N)$ of index r.

Thus $f(M,N)$ is a function of Ramanujan's class with the index r.

We next derive the closure properties of the classes c_1, c_2.

THEOREM 3. *The composite of two functions of c_{1r}, c_{1s} respectively of the class c_1 belongs to c_{1t}, where $t = r$ if $s > r$ and $t \geqslant r$ if $s = r$.*

For, if $f_r(M,N)$, $f_s(M,N)$ belong to c_{1r}, c_{1s} respectively and $c_r(M,N)$, $c_s(M,N)$ are their respective c-components, the c-component of their composite is by [1, 522] the composite of $c_r(M,N)$ and $c_s(M,N)$. Now $\{ c_r(M,N).c_s(M,N) \}$ is an ordinal function whose index is r if $s > r$ and $\geqslant r$ if $s = r$ [2, 95]. Hence the theorem follows.

THEOREM 4. *The inverse of a function of c_{1r} belongs to c_{1r}.*

For, if $F(M,N)$ belongs to c_{1r} and $c_r(M,N)$ is its c-component, $F^{-1}(M,N)$ has the c-component $c_r^{-1}(M,N)$ [1, 522] whose index is precisely r [2, 97].

THEOREM 5. *The composite of two functions of c_{2r}, c_{2s} respectively belongs to c_{2r} if $s = r$; if $s > r$ the composite belongs to c_{1r}.*

For the c-component of the composite of two functions of c_{2r}, c_{2s} respectively is the composite of their c-components and therefore if $s = r$, it is the composite of two quasi-principal functions of the same index r which, by [2, 97], is itself a quasi-principal function of index r. Hence the first part of the theorem follows.

Regarding the second part we note that the composite of two quasi-principal functions of indices r, s $(s > r)$ respectively though, not a quasi-principal function, is an ordinal function of index r.

THEOREM 6. *The inverse of a function of c_{2r} belongs to c_{2r}.*

For, the c-component of the inverse of a function is the inverse of its c-component; hence if a function belongs to c_{2r}, the c-component of its inverse is the inverse of a quasi-principal function of index r, which, by [2, 97] is quasi-principal with the index r.

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ON THE PARITY OF THE PARTITION FUNCTION $p(n)$

BY

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Up till now no simple criterion is known to determine directly the parity of $p(n)$—the number of unrestricted partition of n. As early as 1921, in response to an enquiry from Ramanujan, Macmahon gave a method for ascertaining the parity of $p(n)$. Very recently, Gupta has utilised the parity property of Ramanujan's function $\tau(n)$ to obtain Macmahon's result. The object of this paper is to establish a new method with a similar end in view.

We take $p(0) = 1$ and $p(a) = 0$ if a is not a positive integer.

2. THEOREM. $p(n) \equiv \sum_t p\left[\left\{n-(t-5)/8\right\}/16\right] \pmod 2$;

where t runs through all positive integers of the form $p^\alpha \mu^2$, p being a prime of the form $8m+5$, $\alpha \equiv 1 \pmod 4$ *and* $(\mu, 2p) = 1$.

For,

$$\sum_{n=0}^{\infty} p(n)\, x^n = \prod_{r=1}^{\infty} (1-x^r)^{-16} \prod_{r=1}^{\infty} (1-x^r)^{4.3} \prod_{r=1}^{\infty} (1-x^r)^3$$

$$\equiv \prod_{r=1}^{\infty} (1-x^{16r})^{-1} \prod_{r=1}^{\infty} (1-x^{4r})^3 \prod_{r=1}^{\infty} (1-x^r)^3$$

$$\equiv \left\{\sum_{r=0}^{\infty} (p(r) x^{16r}\right\} \left\{\sum_{i=0}^{\infty} x^{2i(i+1)}\right\} \left\{\sum_{s=0}^{\infty} x^{s(s+1)/2}\right\}$$

$$\equiv \left\{\sum_{r=0}^{\infty} p(r) x^{16r}\right\} \left\{\sum_{i=0}^{\infty} \sum_{s=0}^{\infty} x^{2i(i+1)+s(s+1)/2}\right\}$$

$$\equiv \left\{\sum_{r=0}^{\infty} p(r) x^{16r}\right\} \left\{\sum_{m=0}^{\infty} c(m)\, x^m\right\} \pmod 2, \quad (1)$$

where $c(m)$ denotes the total number of representations of m as $m = 2i(i+1)+s(s+1)/2$ or $8m+5$ as

$$8m+5 = 4(2i+1)^2+(2s+1)^2, \; i, s \geqslant 0.$$

A moment's consideration enables one to identify $c(m)$ with $r(8m+5)/8$, $r(k)$ denoting the number of representations of k as the sum of two squares, order and signs being taken into account.

Our purpose is now to find when $c(m)$ can be odd. In the first place, if $c(m) = r(8m+5)/8 \neq 0$ we must have $8m+5 = \pi p_i^{\alpha_i} \pi q_i^{2\beta_i}$ where, $p_i \equiv 1 \pmod 4$, $q_i \equiv -1 \pmod 4$ and of course at least one p_i say p_1 is of the form $8k+5$ and the corresponding α_1 is odd. We recall that $c(m) = r(8m+5)/8 = \pi_i (\alpha_i+1)/2$. If two or more α's are odd, $c(m)$ is even. Consequently, when $c(m)$ is odd, only α_1 is odd. Besides $\alpha_1 \not\equiv -1 \pmod 4$, otherwise $c(m)$ is even. Thus $c(m)$ is odd if and only if $8m+5 = p_1^{\alpha_1} \mu^2$, where p_1 is a prime of the form $8k+5$, $\alpha_1 \equiv 1 \pmod 4$ and $(\mu, 2p_1) = 1$.

Hence we can write (1) as

$$\sum_{n=0}^{\infty} p(n)\, x^n \equiv \left\{ \sum_{r=0}^{\infty} p(r)\, x^{16r} \right\} \left\{ \sum_{t} x^{(t-5)/8} \right\} \pmod 2,$$

t running through all integers specified above. Comparing coefficients, on both sides we get the theorem.

EXAMPLE. $p(200) \equiv p(4)+p(7)+p(10)+p(11) \equiv 1+1+0+0 \equiv 0 \pmod 2$. It is to be noted that the present formula has the advantage of requiring a knowledge of $p(r)$ only for $r \leqslant n/16$, whereas Macmahon's formula uses the parity of $p(r)$ for r as far as $[n/4]$. On the other hand when n is sufficiently large the number of terms in the above formula is larger than in Macmahon's formula. But for large n recursive formulas are practically useless in view of the heavy and tedious computational labour involved. A factor table would also be necessary in the case of the present formula.

REFERENCES

1. H. GUPTA : A note on the parity of $p(n)$. *Journ. Ind. Math. Soc.*, (2) 10 (1946).

NOTE BY REFEREE. In practice, we need give to t only values of the form :

$128k+8n_0+5$, where $n \equiv n_0 \pmod{16}$, $0 \leqslant n_0 < 16$, $0 \leqslant k \leqslant [n/16]$. We retain only such of these values which are of the specified form.

ON THE STRONG EXTENSION OF A T_1-SPACE INTO A T_1-BICOMPACT SPACE*

BY

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1. ABSTRACT. This paper is generally concerned with irregular bicompact spaces. It is proved that the strong extension of a T_1-space into a T_1-bicompact space by the addition of a single point ξ as described by Alexandroff and Hopf (*Topologie* I, p. 93) is locally minimal-bicompact at the point ξ and it is deduced that every bicompact space is locally minimal-bicompact at every regular point of the space. A general method of obtaining irregular minimal-bicompact spaces is finally proved.

2. DEFINITIONS. A topological space is said to be bicompact, if, from every open covering of the space, we can select a finite covering of the space. A topological space is said to be bicompact *at a point* of it, if the point has a neighbourhood, whose closure, considered as relative space, is bicompact. A topological space is said to be *locally bicompact* if it is bicompact at every point of it. A bicompact space $(R; f)$ is said to be minimal-bicompact if any topology on R, which is weaker than f, renders the space non-bicompact. A bicompact space R will be said to be *locally minimal-bicompact* at a point $p \varepsilon R$, if the the topology of the space cannot be weakened at p without rendering the space non-bicompact.

3.1. THEOREM I. *Any T_1-space can be topologically imbedded in a T_1-bicompact space.*

PROOF. Let R be a T_1-non-bicompact space. Now the union of any two closed bicompact subsets of R is

* My thanks are due to Dr. R. Vaidyanathaswamy, for his valuable help in the preparation of this paper.

itself a closed bicompact subset of R. Further the intersection of any number of closed bicompact subsets of R is also a closed bicompact subset of R. Hence the closed bicompact subsets of R form a Δ-ring $\{S\}$. This ring will not contain the whole space R, since, by hypothesis, R is not bicompact. The complements of the elements of $\{S\}$ form a Σ-ring $\{F\}$ which will not contain the null set. Take any definite sub-ring $\{G\}$ of $\{F\}$ such that the intersection of all the G's is null. It follows that there is a G disjoint with any assigned point of R. Let us now define a topological space $R+\xi$ by the addition of an abstract point ξ, where the neighbourhoods of points of R are the open sets of R containing these points and the neighbourhoods of ξ are prescribed to be sets of the form $\{\xi+G\}$. It is easy to see that Hausdorff's postulates are satisfied in $R+\xi$, and the space $R+\xi$ is therefore a topological space. Since R is a T_1-space and since there is a G disjoint with any prescribed point of R, it follows that $R+\xi$ is a T_1-space in which R is topologically imbedded.

We shall now prove that $R+\xi$ is bicompact. Let $\{g\}$ be any open covering of $R+\xi$ and let $g(\xi)$ be an element of $\{g\}$ containing ξ. Then $[(R+\xi)-g(\xi)]$ is a closed bicompact subset of R and can therefore be covered by a finite number of g's. This finite number of g's together with $g(\xi)$ will cover $(R+\xi)$. Hence $(R+\xi)$ is bicompact.

3.2. Conversely we have

THEOREM 2. *If the T_1-space R be topologically imbedded in the T_1-bicompact space $R+\xi$, then the complement of every neighbourhood of ξ is a closed bicompact subset of R.*

PROOF. If $U(\xi)$ be any neighbourhood of ξ in $R+\xi$ then $[R-U(\xi)]$ is closed in $R+\xi$ and hence is a bicompact subset of $R+\xi$. But $[R-U(\xi)] \subset R$. Hence $[R-U(\xi)]$

is a closed bicompact subset of R. Thus we see that the intersections of the neighbourhoods of ξ with R form a sub-ring of the Σ-ring $\{F\}$ of R.

3·3. COR. It is clear, by Theorem I above, *that any T_1-space can be rendered bicompact by the addition of a single point ξ, where the neighbourhoods of ξ are open sets of the form $\{\xi+G\}$, the family $\{G\}$ being a suitable sub-ring of the Σ-ring $\{F\}$ of R.*

It is also clear that the bicompact space $R+\xi$ will be locally minimal bicompact at ξ only when we choose the entire Σ-ring $\{F\}$ to from the neighbourhoods of ξ. The bicompact space $R+\xi$, in this case, is said to be formed, after Alexandroff and Hopf (*Topologie* I, p. 93), by the *strong extension* of the space R.

3·4. COR. *The locally minimal-bicompactness of $R+\xi$ at the point ξ can be alternatively expressed thus* :—

Let p be any point with the neighbourhood system $\{f(p)\}$ of the bicompact space $(R;f)$. Then the system $\{f(p)\}$ is stronger than or equivalent to the neighbourhood system $\{\phi(p)\}$ of p which would have been obtained by the strong extension of $R-p$ into R.

4. We shall now prove

THEOREM 3. *A bicompact space is locally minimal-bicompact at every regular point of it.*

PROOF. Let $(R;f)$ be a bicompact space and p a regular point of it. With the notation of the corollary (3·4) above, we see that every $f(p)$ is a $\phi(p)$. Now let S be any closed bicompact subset of $R-p$. Then clearly S is bicompact in R also. Now p, being regular in R, is H-separable from every point of S and hence, since S is bicompact, p is H-separable from S itself.* Therefore

* Vide R. Vaidyanathaswamy, *Set Topology*, p. 101.

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p cannot be contact point of S in R. Hence S is a closed bicompact subset of R also. It follows that every $\phi(p)$ is also an $f(p)$. Thus the systems of neighbourhoods $\{\phi(p)\}$ and $\{f(p)\}$ are equivalent.

In other words the given space is locally minimal-bicompact at p.

We deduce immediately that a regular bicompact (i.e. Hausdorff bicompact) space is necessarily minimal-bicompact.

5. THEOREM 4. *In the strong extension of the T_1-space R into the T_1-bicompact space $(R+\xi)$, the point ξ will be regular if and only if the space R is locally bicompact.*

PROOF. To prove the first part, let R be locally bicompact. Then there exists a neighbourhood $f(p)$ of $p \varepsilon R$ such that $\overline{f(p)}$ is bicompact; and we know that $\xi+[R-\overline{f(p)}]$ is a neighbourhood of ξ in $R+\xi$, which is disjoint with the $f(p)$ chosen of p. Thus ξ is H-separable from every point of R. Now take any neighbourhood $U(\xi)$ of ξ and let K be the frontier of $U(\xi)$. Then K, being a closed subset of the bicompact space $R+\xi$, is itself bicompact. Since ξ is H-separable from every point of the bicompact subset K, it is H-separable from K itself. Let $U_1(\xi)$ be a neighbourhood of ξ contained in $U(\xi)$ and disjoint with a neighbourhood $f(K)$ of K, Then

$$0=f(K).U_1(\xi)=f(K).\overline{U_1(\xi)}=K.\overline{U_1(\xi)}.$$

But $\overline{U_1(\xi)}\subset\overline{U(\xi)}=U(\xi)+K$. It follows that $\overline{U_1(\xi)}\subset U(\xi)$. Hence every neighbourhood of ξ contains the closure of another neighbourhood of ξ; i.e. ξ is regular in $R+\xi$.

To prove the second part, let ξ be regular. Then it is H-separable from every point of R. Hence if p be any point of R, there exist an $f(p)$ and a $U(\xi)$ such that $0=f(p).U(\xi)=\overline{f(p)}.\ U(\xi)$. Hence ξ is not contact point

of $\overline{f(p)}$. Now $f(p)$, being a closed subset of the bicompact space $R+\xi$, is itself bicompact. Since ξ is not contact point of $\overline{f(p)}$, it follows that $\overline{f(p)}$ is closed and bicompact in R also. Further $f(p)$ is a neighbourhood of p in R. Hence R is bicompact at p, and hence is locally bicompact.

We deduce immediately that ξ will be *irregular* if and only if R is not bicompact at least at one point. The point ξ will then be H-inseparable from those points of R at which R is not bicompact.

5. Note. In the course of the proof of the first part of Theorem 4, we have seen that if a point of a bicompact space is H-separable from every other point of the space then the point is necessarily regular. We conclude therefore that every point of a bicompact space is either regular or H-inseparable from at least one other point of the space. Hence, in an irregular bicompact space, there must exist at least one pair of H-inseparable points. We may now theoretically establish the existence of irregular minimal-bicompact spaces.

Consider a bicompact space $(R;f)$ containing a pair of H-inseparable points p and q; and such that $(R-p)$ is not bicompact at q and $(R-q)$ is not bicompact at p. Let us obtain the bicompact space $(R;\phi)$ by the strong extension of $(R-p)$ into R. In the ϕ-topology, p and q are H-inseparable and $(R-q)$ is not bicompact at p. Let us now obtain the bicompact space $(R;\phi_1)$ by the strong extension of $(R-q)$ into R. The points p and q will still be H-inseparable; and the space $(R;\phi_1)$ is locally minimal-bicompact at p and q. Repeating the above process of strong extension with respect to every other point of R, we finally arrive at a minimal bicompact space $(R;\psi)$ in which at least p and q are H-inseparable and which is therefore an irregular minimal-bicompact space.

The existence of spaces satisfying the conditions required for the above argument has been demonstrated already.*

* "Minimal-bicompact spaces", *Journal of the Indian Mathematical Society*, (2) (1947).

The space in the above paper has been obtained from the space R constructed by Urysohn for a different purpose (*Mathematische Annalen* Band 94, p. 268), in the following manner. The strong extension of R into $R+\xi$ is first obtained and ξ is then topologically identified with the point a. It is easily proved that the space R after topological identification of ξ with a becomes an irregular bicompact space satisfying the conditions required for the above argument. It will be of interest to note that the space $R+\xi$ is itself minimal-bicompact and therefore also R in the strengthened bicompact topology.

PRIME IDEALS AND THE THEORY OF LAST-RESIDUE-CLASSES*

BY

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INTRODUCTION. The notion of prime ideals in a distributive lattice was first introduced and systematically developed by Stone[2], who utilized this powerful tool for his representations of a distributive lattice as a ring of sets. A fundamental result of Stone's theory is that every ideal (except the whole lattice) is the product of all its prime ideal divisors. The theory of last-residue-classes in a distributive lattice, first studied by Krishnan,[1] may be elegantly developed with the help of prime ideals. In this paper we exploit the concept of the prime ideal to effect a solution of the following three problems in the last-residue-class theory proposed by Vaidyanathaswamy, for the first two of which a solution has already been given by Krishnan in a different way :—

PROBLEM 1—*to characterize the l.r.c. (last-residue-class) of an ideal P of L† by means of ideal operations on P;*

PROBLEM 2—*to determine the conditions under which an ideal is the l.r.c. of another;*

PROBLEM 3—*to determine the ideals of L which are identical with the l.r.c. of their l.r.c.*

The main tools of Krishnan's investigation are "product-complementation" and "ideal-extension" whereas, the corner-stone of our solution is the concept of the prime ideal.

For an excellent exposition of the lattice theoretic notions used in this paper we refer to Vaidyanathaswamy.[4] We shall state all our definitions and theorems only for the case of α-ideals; the dual formulation of these for μ-ideals will be obvious.

* I am grateful to Dr. R. Vaidyanathaswamy for his help in bringing out this paper.

† *L* denotes the given distributive lattice.

1. *Congruences and Last-Residue-classes.*

Let L be a distributive lattice with units (0, 1). *Congruence* with respect to an α-ideal A_α in L is defined by: $x \equiv y \pmod{A_\alpha}$ if there exist elements s, t in A_α such that $x\,s = y\,t$. It is easy to see that this *congruence* is an *equivalence-relation* partitioning L into disjoint residue-classes (x and y belonging to the same residue-class, if and only if, $x \equiv y \pmod{A_\alpha}$, and that A_α itself is one of the residue-classes namely, the residue-class containing the element 1). The residue-class of A_α containing the element 0 is called the last-residue-class or l. r. c. of A_α; it can be shown that the l.r.c. of an α-ideal is a μ-ideal, and vice-versa. A detailed investigation on l.r.c. will be found in Krishnan's paper. We give below a few simple properties relating to last-residue classes which we require in the sequel. The proofs are omitted.

1.1. (1) $x \equiv 0 \pmod{A_\alpha}$, if and only if, there exist $p \,\varepsilon\, A_\alpha$ with $x\,p = 0$.

(2) If A_α is contained in B_α, the l.r.c. of A_α is contained in the l.r.c. of B_α.

(3) The l.r.c. of the product of a *finite* number of ideals is the product of their l.r.c.'s.

(4) Every ideal contains the l.r.c. of its l.r.c.

(5) An ideal A_α is said to be semi-simple if $A_\alpha + P'_\alpha(x)^* = 1$ † for each $x \,\varepsilon\, A_\alpha$. The semi-simple ideals are precisely, those that are the l.r.c. of their l.r.c.‡

2. *Prime and Divisorless Ideals.*

DEFINITION 1. *An ideal $A_\alpha \neq 1$ is said to be prime if it contains with the sum $x+y$ of two elements x, y one at least of x, y.*

2.1. It is easy to see that a necessary and sufficient condition for A_α to the prime is that the complement

* $P_\alpha(x)$ denotes the principal α-ideal defined by x, i.e. the set of elements $y > x$; dually, $P_\mu(x)$ is the set of elements $y < x$.

† Here 1 denotes the whole lattice considered as an α-ideal.

‡ For proof, see § 2 of (3)

$c\,(A_\alpha)$, i. e. the set of elements of L not contained is A_α constitute a μ-ideal; also, $c\,(A_\alpha)$ is prime whenever A_α is prime.

2.2. Since two residue-classes cannot intersect without being identical, it follows that every ideal $\neq 1$ is disjoint with its l.r.c.; in particular, this is true of prime ideals. Consequently, we have by §2.1 complements of prime factors of an ideal are prime factors of its l.r.c.

Definition 2. *An ideal $M_\alpha \neq 1$ is said to be divisorless if the only ideals containing it are itself and* 1.

2.3. Since by definition, a divisorless ideal M_α is such that $M_\alpha + P_\alpha(x)^* = 1$, if and only if, $x \varepsilon c\ (M_\alpha)$, it follows that the divisorless ideals are precisely those ideals whose complements coincide with their l.r.c.'s. Thus, $c(M_\alpha)$ is an ideal, and it therefore follows that a divisorless ideal must be prime.

Definition 3. *A prime ideal M_α is said to be minimal-prime if it contains no prime ideal different from itself.*

2.4. It may be immediately seen that the minimal-prime ideals are identical in their ensemble with the complements of divisorless ideals. For, we observe that by virtue of Stone's factorization† theorem, an ideal is divisorless, if and only if, it has no prime factors different from itself. Thus "M_α is divisorless" is equivalent to "$c\,(M_\alpha)$ does not contain any prime ideal other than itself (§2.1)", i.e. "$c\,(M_\alpha)$ is minimal-prime".

Theorem 1. *A divisorless ideal A is also minimal-prime, if and only if, A is semi-simple.*

Proof. Suppose A_α is both divisorless and minimal-prime. Then by §§2.3, 2.4 each of A_α, $c(A_\alpha)$ is the l.r.c. of the other, or A_α is semi-simple (§1.1). Conversely, if A_α is divisorless and semi-simple, $A_\alpha =$ l.r.c. of (l. r. c. of A_α) on account of semi-simplicity of A_α. Again,

* See § 1.1 footnote

† Every ideal 1 is the product of all its prime ideal divisors. See (2).

since by hypothesis A_α is divisorless, l.r.c. of $A_\alpha = c\ (A_\alpha)$. Hence, $A_\alpha =$ l.r.c. of $c(A_\alpha)$, or A_α is minimal-prime.

The characteristic property of prime ideals is embodied in

THEOREM 2. (THEOREM OF STONE). *A prime ideal cannot contain the intersection of a finite number of ideals without containing at least one of them.**

PROOF. For, if possible, let A_α contain the intersection of two ideals B_α, C_α without containing either of them. Then there exist elements $b \varepsilon B_\alpha$, $b \bar{\varepsilon} A_\alpha$; $c \varepsilon C_\alpha$, $c \bar{\varepsilon} A_\alpha$, while $b+c \varepsilon A_\alpha$: this contradicts the definition of prime ideals, whence the result is established for the case of two ideals. By induction this can be extended to any finite number of ideals.

THEOREM 3. (DUAL OF STONE'S THEOREM). *If a prime α-ideal A_α intersects the ideal sum ΣB^i_μ of any number of μ-ideals, it must intersect some B^i_μ (say, for $i = j$).*

PROOF. Let the prime ideal A_α contain the element x belonging to ΣB^i_μ. By definition of the ideal-sum,

$$x = x_1 + x_2 + \ldots + x_m + \ldots + x_n \quad (x_m \varepsilon P^m_\mu).$$

Since A_α contains $x = x_1 + x_2 + \ldots + x_n$ and is prime, it must contain some x_i (say, for $i = j$) (Theorem 2). Thus A_α intersects P^j_μ in x_j, whence the theorem.

4. EXISTENCE THEOREMS.

Stone's existence theorem on prime ideals(2) states: *Given any two distinct ideals (each different from 1), there exists a prime ideal containing one but not the other.*

We formulate and prove below an existence theorem for divisorless ideals (which is possibly known) in a form convenient for our purpose.

THEORM 4. *Given any arbitrary ideal $A_\alpha \neq 1$, there exists a divisorless ideal M_α containing A_α; further M_α is disjoint with the l.r.c. of A_α.*

PROOF. We may appeal to Zorn's lemma or the following form of it which will suffice for our purpose

* Compare Theorem 8 of (2).

The elements below which the complete lattice L_α (L_μ) of α-(μ-) ideals of a distributive lattice displays a jump, are precisely the principal ideals ; in particular, the unit is a jump element of L_α (L_μ)†.*

Let A_α be any ideal $\neq 1$. Take $A_\alpha = A_1$. If A_1 is not divisorless it has a divisor $A_2 \neq 1$; if A_2 is not divisorless it has a divisor $A_3 \neq 1$; and so on. Assuming that for any integer n we have thus constructed the ideals $A_1, A_{21} \ldots A_n$ different from each other and from 1, we can take $A_{n+1} \neq 1$ as a proper divisor of A_n, if A_n is not divisorless. More generally, assuming that for all ordinals $\xi < \eta$ we have constructed ideals A_ξ different from each other and from 1 with $A_\lambda \subset A_\mu$ for $\lambda < \mu < \eta$, we can define

$$A_\eta = \text{a proper divisor of } A_{\eta-1}, \text{ if } \eta \text{ is non-limiting}$$
$$= \sum_{\xi < \eta} A_\xi, \text{ if } \eta \text{ is limiting.}$$

Assuming that A_α is not contained in any divisorless ideal, there will be a first ordinal γ such that $A_\gamma = 1$. From the mode of our construction γ must be limiting. Thus we have a well-ordered increasing sequence of α-ideals ending with 1, in which 1 is not a jump-element. This contradicts the fact that the ideal 1 is a jump-element of L_α. Hence the first part of the result. The second part follows from § 2.2.

Cor. *Every prime ideal is contained in a divisorless ideal and contains a minimal-prime ideal.*

The first part follows from Theorem 4 since a prime ideal $\neq 1$. To prove the second part we observe that if A_α is prime, $c(A_\alpha)$ is a prime μ-ideal (2.1) and hence contained in a divisorless μ-ideal M_μ; consequently, $A_\alpha = cc(A_\alpha) \supset c(M_\mu)$; since M_μ is divisorless, $c(M_\mu)$ is minimal-prime, whence the result.

5. We now proceed to characterize the l.r.c. of an ideal. For this purpose we require

* i.e. ideals of the form $P_\rho(a)$ or $P_\mu(a)$; see 1.1.

† See 12.62 of (4).

THEOREM 5. *The last-residue-class of an ideal A is the intersection of the complements of the prime factors of A.*

PROOF. Let (A^i_α) be the prime factors of the ideal $A = A_\alpha$ Since complements of prime factors of an ideal are prime factors of its l.r.c. (2.2), it follows that $\pi c(A^i_\alpha)$ contains $B_\mu =$ l.r.c. of A_α. If $\pi c(A^i_\alpha) \neq B_\mu$ there exists an element x such that $x \varepsilon c(A^i_\alpha)$, for each i, but $x \bar{\varepsilon} B_\mu$. Further $A_\alpha + P_\alpha(x) \neq 1$: for otherwise $xy = 0$ for some $y \varepsilon A_\alpha$, or $x \varepsilon B_\mu$, contradicting the choice of x. By our existence theorem (Theorm 4), there is a divisorless (and therefore also prime) ideal $M_\alpha \supset A_\alpha + P_\alpha(x)$. Since M_α contains x, it is different from each A^i_α; this contradicts our assumption that the (A^i_α)'s comprise all the prime factors of A_α, whence the theorem is proved.

REMARK. It may be noted that in general the complements of prime factors of A_α do not completely exhaust the prime factors of $B_\mu =$ l.r.c. of A_α; in other words, there may be prime factors B_μ which intersect A_α. In fact, this is always the case except when A_α is semi-simple. For, if A_α is not semi-simple, A_α is not the l.r.c. of B_μ, and so there exists $x \varepsilon A_\alpha$ such that $B_\mu + P_\mu(x) \neq 1$; any prime factor of $B_\mu + P_\mu(x)$ is a prime factor of B_μ intersecting A_α in x.

THEOREM 6. *The largest ideal whose l.r.c. = l.r.c. of a given ideal A, is the intersection πM^i of all divisorless ideals containing A.*

PROOF. Let πM^i_α be the intersection of all divisorless α-ideals containing A_α. First observe that by (2) of § 1.1 $C_\mu =$ l.r.c. of πM^i_α contains $B_\mu =$ l.r.c. of A_α. Suppose $C_\mu \neq B_\mu$; then there exists x contained in C_μ but not, in B_μ. Further each M^i_α being disjoint with its l.r.c. cannot contain x. Since by our supposition $A_\alpha + P_\alpha(x) \neq 1$ there is a divisorless ideal M_α containing $A_\alpha + P_\alpha(x)$ (Theorem 4). Clearly M_α is a divisorless factor of A_α containing x, and hence different from each M^i_α. Thus we arrive at a contradiction. This proves that $C_\mu = B_\mu$.

Now let D_α be any ideal whose l.r.c. $= B_\mu$. If an element x of D_α does not belong to πM^i_α, x must belong to some $C(M^i_\alpha)$ (say, for $i=j$). Since M^j_α is divisorless, $xy=0$ for some $y \varepsilon M^j_\alpha$. Thus y lies in the l.r.c. of $D_\alpha = B_\mu \subset$ l.r.c. of M^i_α, since $A_\alpha \subset M^j_\alpha$. Therfore M^j_α intersects its l.r.c. in the element y, which is a contradiction. So we should have $D_\alpha \subset \pi M^i_\alpha$, and the proof is complete.

Since every divisorless α-ideal contains the ideal $0 = P_\alpha(1)$, and since the l.r.c. of $P_\alpha(1) = P_\mu(0)$, we obtain the

Cor. *The largest α-ideal whose l.r.c. $= P_\mu(0)$ is the intersection of all the divisorless α-ideals of the given lattice.*

We deduce the following characterization of the l.r.c. of an α-ideal :

Theorem 7. *The l.r.c. of an α-ideal A_α is the intersection of all minimal-prime μ-ideals M_μ disjoint with A_α.*

Proof. This is an immediate consquence of Theorems 5 and 6, and the obvious fact that the minimal prime μ-ideals disjoint with A_α are precisely, the complements of the divisorless factors of A_α.

A parallel theorem holds for μ-ideals, which may be proved similarly.

Thus, Theorem 7 completely solves Problem 1. Now we give solutions to Problems 2 and 3 (Theorems 8, 9).

Theorem 8. *A necessary and sufficient condition that an α-ideal A_α is the l.r.c. of some μ-ideal B_μ is the existence of a set of divisorless μ-ideals (M_μ) whose set-union is the set-complement of A_α, and which exhaust all the divisorless ideal factors of their intersection.*

Proof. That the condition is necessary, is a consquence of Theorem 7. For, if $A_\alpha =$ l.r.c. of B_μ, the divisorless factors M_μ of B_μ satisfy the conditions-(i) $\cup M_\mu{}^* = c(A_\alpha)$, (ii) every divisorless factor of $\cup M_\mu$ is disjoint with A_α. Conversely, if (M_μ) is a family of divisorless

* $\cup M_\mu$ denotes the *set-union* of thc family of ideals (M_μ).

μ-ideals satisfying the above conditions w.r.t. the ideal A_α, it is easy to see that $A_\alpha =$ l.r.c. of $_\pi M_\mu$. Hence the proof is complete.

THEOREM 9. *An α-ideal $A_\alpha \neq 1$ is semi-simple, if and only if, every divisorless factor of the l.r.c. of A_α is disjoint with A_α.*

PROOF. For, if A_α is semi-simple, $A_\alpha =$ l.r.c. of B_μ where $B_\mu =$ l.r.c. of A_α. Hence, by §2.2 every divisorless factor of A_α is disjoint with B_μ. For the converse, we observe that if A_α is not semi-simple, $A_\alpha \neq$ l.r.c. of B_μ. Consquently, there exists $x \in A_\alpha$ such that $B_\mu + P_\mu(x) \neq 1$. Any divisorless ideal containing $B_\mu + P_\mu(x)$ is a divisorless factor of B_μ intersecting A_α in x.

The analogues of Theorems 8 and 9 for μ-ideals may be proved by similar methods.

It might be thought from the form of Theorem 8, that the intersection of any family of minimal-prime ideals is the l.r.c. of some ideal. That this is false, is shown by the following

EXAMPLE. Γ is the lattice of open-sets of the topological space R consisting of the points (0); $(1/n)$, $n = 1, 2, \ldots$ The topology of R is the relative topology of the set R considered as a subset of the real-number-space. It is easy to see that the sets $(g_n) = (1/n)$, $n = 1, 2, \ldots$ are all open in R. Moreover, we can show that the ideals $P_\alpha(g_n)$ of Γ are all divisorless and simple,* and hence also minimal-prime (Theorem 1). Now we shall show that $\prod_{n=1}^{\infty} P_\alpha(g_n) = P_\alpha(g)$, where $g = \sum_{1}^{\infty} g_n$, is not the l.r.c. of any ideal. For, if possible, let $P_\alpha(g) =$ l.r.c. of A_μ. Since g belongs to the l.r.c. of A_μ there exists $x \in A_\mu$ such that $g + x = 1$. Further x is an open-set containing the point 0. Since any open-set containing

* An ideal A is said to be simple if $A + A' = 1$; obviously every simple ideal is semi-simple.

o is also closed, the complement $c(x)$ of x must belong to $P_\alpha(g)$. But, since g is a maximal-element* of Γ, this is clearly impossible, whence our supposition is untenable.

However, the above statement would be true provided the family of ideals is finite. For by (3) of §§1.1 and 2.4, we have $\prod_{i=1}^{n} m_\alpha^i = $ l.r.c. of $\prod_{i=1}^{n} M_\mu^i$, where $M_\mu^i = c(m_\alpha^i)$, $i = 1, 2 \ldots n$.

6. *Necessary and sufficient conditions for a Distributive Lattice to be a Boolean-algebra.*

THEOREM 10. *Any one of the following four equivalent conditions is necessary and sufficient for a distributive lattice with (o, 1) to be a Boolean-algebra.*

(1) Every prime α-ideal is divisorless.
(2) Every prime μ-ideal is divisorless.
(3) Every prime α-ideal is minimal-prime.
(4) Every prime μ-ideal is minimal-prime.

It follows from § 2.4 that the conditions (1) and (4) are equivalent, as also (2) and (3). To show that conditions (1) and (2) are equivalent, it suffices to remark that if A_μ is a prime μ-ideal with a proper prime divisor B_μ, $c(B_\mu)$ is a prime α-ideal with $c(A_\mu)$ as one of its proper prime-divisors. Thus the equivalence of the given set of conditions has been established. Further, since in a Boolean algebra the equations $x+x' = 1$, $x.x' = 0$ hold for each x, a prime α-ideal A_α must necessarily contain one and only of x, x'. Hence, $A_\alpha + P_\alpha(x) = 1$ for each $x \varepsilon c(A_\alpha)$, or A_α is divisorless†. This proves that the above conditions are necessary.

For proving the sufficiency of the given conditions we need the following

* An element $m \neq 1$ of a distributive lattice with 1 is said to to be maximal if $x > m$ implies $x = m$ or 1.

† Compare M. H. Stone, 'The Theory of Representations for Boolean Algebras,' *Trans. Amer. Math. Soc.* Theorem 33. 40 (1936).

LEMMA. *The non-minimal prime ideals are identical in their totality, with the prime factors of ideals of the form* $P(x) + P'(x)$*.

Suppose A_α is a non-minimal prime α-ideal so that it contains a prime ideal B_α with $B_\alpha \neq A_\alpha$; let $x \varepsilon A_\alpha, x \bar{\varepsilon} B_\alpha$. Since $P_\alpha(x). P'_\alpha(x) = 0$, $P_\alpha(x) \not\leq P_\alpha$, it follows from Theorem 2 that $P'_\alpha(x) \subset B_\alpha \subset A_\alpha$. Consequently, A_α is a prime factor of $P_\alpha(x) + P'_\alpha(x)$. Conversely, we have to show that every prime factor of an ideal of the form $P_\alpha(x) + P'_\alpha(x)$ is non-minimal. For, if possible, let m_α be a minimal-prime ideal containing the ideal $P_\alpha(x) + P'_\alpha(x)$. Since $m_\alpha =$ l.r.c. of $c(m_\alpha)$ (§§ 2.3 and 2.4), we can find $y \varepsilon c(m_\alpha)$ such that $x + y = 1$, i.e. y belongs to $P'_\alpha(x)$. Hence we reach the contradiction that an ideal $\neq 1$ intersects its l.r.c., whence the result.

It is clear that in every distributive lattice L which is not a Boolean-algebra $P_\alpha(x) + P'_\alpha(x) \neq 1$ for some $x \varepsilon L$. By the above lemma, no prime factor of $P_\alpha(x) + P'_\alpha(x)$ can be minimal prime. Thus, condition (3) of the hypothesis is sufficient, and on account of the equivalence of the conditions (1), (2), (3) and (4) each of the other conditions is also sufficient.

* $P(x)$ denotes the principal ideal corresponding to x.

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ZETA FUNCTIONS ON THE SPHERE*

BY

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A large number of Dirichlet's series with finite abscissæ of absolute convergence have been discovered, which display properties in common with those of Riemann zeta series. Epstein, Ramanujan, Hecke, Siegal, and Weil are some of the mathematicians who have discovered the most important classes of such Dirichlet's series. Most of them are such that they are either entire functions with the so called 'trivial zeros' at negative integers or meromorphic functions with simple poles at some points on the real axis and with trivial zeros at negative integers. A few months ago the author had observed still another class of Dirichlet's series, which may be viewed as a generalization of Epstein zeta functions.† They arise thus:

Consider the eigenvalues and eigenfunctions of the following problem

$$\triangle u + \lambda u = 0$$

$$u = 0 \text{ or } \frac{\partial u}{\partial n} = 0 \text{ on } B, \qquad (1)$$

where B is the boundary of a bounded Euclidean domain D of k dimensions and Δ is the Laplace operator, $\sum_{i=1}^{k} \frac{\partial^2 u}{\partial x_i^2}$. If $\omega_n(x)$ are the eigenfunctions with the non-negative eigenvalues λ_n arranged in non-decreasing order of magnitude, then the two Dirchlet's series

* The substance of this paper was read in Feb. '48 in the seminar on Zeta functions organised by Profs. C. L. Siegal and H. Weyl at the Institute for Advanced Study, Princeton.

† S. Minakshisundaram: 'A Generalization of Epstein Zeta Functions' to appear in the *Canad. Jour. Math.*

$$\frac{\sum_{1}^{\infty} \omega_n(x)\, \omega_n(y)}{\sum_{1}^{\infty} \frac{\omega_n^2(x)}{\lambda_n^s}} \qquad \lambda_1 > 0,\ x \text{ and } y \text{ interior points} \qquad (2)$$

have properties similar to those of Epstein zeta functions. A further generalization will be obtained if we consider a Riemannian manifold instead of an Euclidean domain. Pleijel and I have considered this case elsewhere.*

The aim of this note is to study the Dirichlet's series arising in the special case of a sphere and give a direct and simple proof of the results already proved in the general case, using known properties of harmonics on the sphere.

1. In this section I enumerate the known properties of harmonics on the sphere† and of Bessel functions with imaginary arguments,‡ which will be used in the following sections.

HARMONICS ON THE SPHERE. Harmonics on the sphere $x_1^2+x_2^2+..+x_{k+1}^2 = 1$ are the solutions of

$$\sum_{i=1}^{k+1} \frac{\partial^2 u}{\partial x_i^2} = 0. \qquad (3)$$

A harmonic of order n is a homogeneous polynomial of degree n in the $k+1$ variables $(x_1, x_2, \ldots, x_{k+1})$ satisfying the equation (3). If we introduce polar coordinates $x_1 = \cos\theta_1, x_2 = \sin\theta_1 \cos\theta_2, \ldots, x_{k+1} = \sin\theta_1 \sin\theta_2 \ldots \sin\theta_k$, the harmonics on the sphere satisfy the equation

$$\wedge u + n(n+k-1)\, u = 0, \qquad (4)$$

where

* S. Minakshisundaram and A. Pleijel: 'Some Properties of the Eigenfunctions of the Laplace Operator on Riemannian Manifolds', to appear in the *Canad. Jour. Math.*

† Heine: *Kügel Funktionen.*

‡ G. N. Watson: *A Treatise on Bessel Functions.*

$$\wedge u = \sum_{r=1}^{k} \frac{\omega_r}{\sin^{k-r}\theta_r} \frac{\partial}{\partial\theta_r}\left(\sin^{k-r}\theta_r \frac{\partial u}{\partial\theta_r}\right)$$

$$\omega_1 = 1,\ \omega_r = (\sin\theta_1 \ldots \sin\theta_{r-1})^2,\ r > 1.$$

In other words the harmonics are the eigenfunctions of

$$\wedge u + \lambda u = 0$$

with the eigenvalues $n(n+k-1)$. The number of independent solutions of (4) is l, where

$$l = \frac{(k+n-2)\,!}{n\,!\,(k-1)\,!}(k+2n-1). \qquad (5)$$

If the fundamental solutions of (4) are $F_i^n(x_1 \ldots x_{k+1})$, $i = 1, 2, \ldots, l$, with

$$\int F_i^n F_j^n = \delta_{ij},$$

then

$$\sum_{i=1}^{l} F_i^n(x_1 \ldots x_{k+1}) F_i^n(y_1 \ldots y_{k+1}) = C_n^k(\cos\theta) . \frac{\Gamma(k+1)/2}{2(\sqrt{\pi})^{k+1}}, \qquad (6)$$

where $\cos\theta = \Sigma x_i y_i$ and C_n^k is the associated Legendre polynomial, which is a solution of

$$\frac{1}{\sin^{k-1}\theta} \frac{d}{d\theta}\left(\sin^{k-1}\theta \frac{dP}{d\theta}\right) + n(n+k-1)P = 0. \qquad (7)$$

Thus

$$C_n^1(\cos\theta) = 2\cos n\theta$$
$$C_n^2(\cos\theta) = (2n+1)\ P_n(\cos\theta)$$
$$C_n^2(\cos\theta) = \frac{(n+1)\sin(n+1)\theta}{\sin\theta}.$$

$C_n^k(\cos\theta)$ satisfies the following property

$$C_n^k(\cos\theta) = O(n^{k-1}). \qquad (8)$$

We need the following known lemma which we take for granted:

LEMMA 1.

$$\sum_{n=0}^{\infty} C_n^k(\cos\theta)\, z^n = \frac{1-z^2}{(1-2z\cos\theta+z^2)^{(k+1)/2}}$$

so that

$$C_n^k(1) = \textit{coefficient of } z^n \textit{ in } \frac{1+z}{(1-z)^k}$$
$$= \frac{(k+n-2)\,!}{n\,!\,(k-1)\,!}\,(k+2n-1) = l.$$

BESSEL FUNCTIONS WITH IMAGINARY ARGUMENTS.

We set

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\left(1 + \frac{z^2}{2(2\nu+2)} + \frac{z^4}{2.4.(2\nu+2)\,(2\nu+4)} + \dots\right)$$
$$= \sum_{m=0}^{\infty} \frac{(z/2)^{\nu+2m}}{m\,!\,\Gamma(\nu+m+1)} = e^{-\nu i\pi/2}\,J_\nu(iz), \tag{9}$$

where J_ν is the Bessel function of order ν. If z is real

$z^\nu I_\nu(z)$ $\begin{cases} \text{is an even function of } z \text{ if } \nu \text{ is an integer,} \\ \text{is an odd function of } z \text{ if } \nu \text{ is half an odd integer.} \end{cases}$ (10)

$$I_\nu(ze^{mi\pi}) = e^{m\nu i\pi}\,I_\nu(z), \; m \text{ an integer.} \tag{11}$$
$$I_n(z) = I_{-n}(z), \; n \text{ an integer.} \tag{12}$$

LEMMA 2.

$$\frac{1}{n^s(n+2\nu)^s} = \left(\frac{2}{\nu}\right)^{s-\frac{1}{2}} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(2s)} \int_0^\infty e^{-(n+\nu)t}\, t^{s-\frac{1}{2}}\, I_{s-\frac{1}{2}}(\nu t)\, dt.$$

PROOF.

$$\frac{1}{n^s(n+2\nu)^s} = \left\{(n+\nu)^2 - \nu^2\right\}^{-s} = (n+\nu)^{-2s}\left\{1 - \frac{2\nu}{(n+\nu)^2}\right\}^{-s}$$
$$= \frac{1}{(n+\nu)^{2s}} \sum \frac{s(s+1)\,.\,.\,(s+r-1)}{r\,!}\,\frac{\nu^{2r}}{(n+\nu)^{2r}}$$
$$= \sum \frac{s(s+1)\,.\;(s+r-1)}{\Gamma(r+1)\,\Gamma(2r+2s)} \int_0^\infty e^{-(n+\nu)t}\, t^{2r+2s-1}\, dt.$$
$$= \left(\frac{2}{\nu}\right)^{s-\frac{1}{2}} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(2s)} \int e^{-(n+\nu)t}\, t^{s-\frac{1}{2}}\, I_{s-\frac{1}{2}}(\nu t)\, dt.$$

2. Using the above properties of harmonics and of Bessel functions we are now in a position to study the problem of analytic continuation of the following two series

$$\sum_{n=1}^{\infty} \frac{C_n^k(\cos\theta)}{n^s(n+2\nu)^s} \tag{13}$$

and

$$\sum_{n=1}^{\infty} \frac{C_n^k(1)}{n^s(n+2\nu)^s}, \tag{14}$$

which are the analogues of (2) in the case of the sphere. On account of the order relation (8), both the series are absolutely convergent for $R(s) > k/2$ and hence represent analytic functions in this half plane. To continue them analytically to the left of this half plane we first observe that the two series can be given infinite integral representations in this region, on using Lemmas 1 and 2. In fact if $\nu = (k-1)/2$ and $R(s) > k/2$

$$\sum_1^{\infty} \frac{C_n^k(\cos\theta)}{n^s(n+2\nu)^s}$$

$$= \left(\frac{2}{\nu}\right)^{s-\frac{1}{2}} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(2s)} \int_0^{\infty} \sum_1^{\infty} C_n^k(\cos\theta)\, e^{-(n+\nu)t}\, t^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(\nu t)\, dt$$

$$= \left(\frac{2}{\nu}\right)^{s-\frac{1}{2}} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(2s)} \int_0^{\infty} \left(\frac{1-e^{-2t}}{(1-2\cos\theta e^{-t}+e^{-2t})^{\nu+1}} - 1 \right) \times t^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(\nu t)\, dt. \tag{15}$$

Similarly if $\theta = 0$, we observe that

$$\frac{1-e^{-2t}}{(1-2e^{-t}+e^{-2t})^{(k+1)/2}} = \frac{1+e^{-t}}{(1-e^{-t})^k} = e^{\nu t} \frac{\cosh t/2}{(\sinh t/2)^k}$$

so that

$$\sum_1^{\infty} \frac{C_n^k(1)}{n^s(n+2\nu)^s} = \left(\frac{2}{\nu}\right)^{s-1/2} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(2s)} \int_0^{\infty} \left\{ \frac{\cosh t/2}{(\sinh t/2)^{-k}} - e^{-\nu t} \right\} \times t^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(\nu t)\, dt. \tag{16}$$

3. Starting from the above integral representations I propose to prove the following theorems :

THEOREM 1. *The series* (13) *can be continued analytically throughout the s-plane and represents an entire function with trivial zeros at* $s = -1, -2, \ldots$

THEOREM 2. *The series* (14) *can be continued analytically throughout the s-plane and represents a meromorphic function with simple poles at*

(*i*) $s = 1, 2, \ldots, k/2$, *if k is even*;

(*ii*) $s = k/2, k/2-1, k/2-2, \ldots$, *if k is odd.*

In case (ii), the function has zeros at $s = -1, -2, -3, \ldots$.

PROOF OF THEOREM 1. In the complex t-plane make a cut along the positive side of the real axis from the origin and take a contour C starting from ∞ in the upper half of the plane, going round the origin in the anti-clockwise direction and then going back to ∞ in the lower half of the plane. The contour C is chosen as to exclude the zeros of $1-2\cos\theta e^{-t}+e^{-2t}$ (except of course the origin which is a zero if $\theta = 0$). Now consider the integral along C, viz.

$$\frac{1}{2\pi i}\int_c \left\{\frac{1-e^{-2t}}{(1-2\cos\theta e^{-t}+e^{-2t})^{(k+1)/2}} - 1\right\} \times e^{-\nu t}(-t)^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(-\nu t)\, dt \qquad (17)$$

$$= e^{-i\pi(2s-1)}\frac{1}{2\pi i}\int_\infty^0 \left\{\frac{1-e^{-2t}}{(1-2\cos\theta e^{-t}+e^{-2t})^{(k+1)/2}} - 1\right\} \times e^{-\nu t}\, t^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(\nu t)\, dt$$

$$+ \frac{1}{2i\pi}\int_. \quad ,,$$

$$+ e^{i\pi(2s-1)}\frac{1}{2i\pi}\int_0^\infty \quad ,,.$$

The second integral on the right along the circle centre at the origin is zero, since the integrand is regular in the circle, and therefore for $R(s) > k/2$, the contour integral (17) is equal to

$$\frac{\sin(2s-1)\pi}{\pi}\int_0^\infty \left\{\frac{1-e^{-2t}}{(1-2\cos\theta e^{-t}+e^{-2t})^{(k+1)/2}} - 1\right\} \times e^{-\nu t}\, t^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(\nu t)\, dt.$$

Thus

$$\sum_1^\infty \frac{C_n^k(\cos\theta)}{n^s(n+2\nu)^s} = -\left(\frac{2}{\nu}\right)^{s-\frac{1}{2}}\frac{\Gamma(s+\frac{1}{2})\,\Gamma(1-2s)}{2i\pi}$$

$$\times\int_c e^{-\nu t}(-t)^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(-\nu t)\left(\frac{1-e^{-2t}}{(1-2\cos\theta e^{-t}+e^{-2t})^{(k+1)/2}} - 1\right) dt. \qquad (18)$$

Since the contour integral is finite for all values of s, it represents a regular function of s. Thus the series on the left of (18) can be continued analytically throughout the plane with the possible exception of the simple poles of $\Gamma(s+\frac{1}{2})\,\Gamma(1-2s)$, viz. half odd integral values and positive integral values of s. But on using (10) and (12), one observes that the integrand is a regular function of t in C, and hence the integral vanishes, thus cancelling the above poles. This shows that the Dirichlet's series represents an entire function of s.

To determine the trivial zeros of the function, the integral is split into two, thus:

$$-\left(\frac{2}{\nu}\right)^{s-\frac{1}{2}}\frac{\Gamma(s+\frac{1}{2})\,\Gamma(1-2s)}{2i\pi}$$

$$\times\int_c e^{-\nu t}\,(-t)^{s-\frac{1}{2}}I_{s-\frac{1}{2}}(-\nu t)\frac{1-e^{-2t}}{(1-2\cos\theta e^{-t}+e^{-2t})^{(k+1)/2}}\,dt$$

$$+\left(\frac{2}{\nu}\right)^{s-\frac{1}{2}}\frac{\Gamma(s+\frac{1}{2})\,\Gamma(1-2s)}{2i\pi}\int_c e^{-\nu t}(-t)^{s-\frac{1}{2}}\,I_{s-\frac{1}{2}}(-\nu t)\;dt.$$

An easy calculation of the residue of the second integral on the right, at $t=0$, shows that it is equal to zero when s is a negative integer. That the residue at $t=0$, when s is a negative integer, of the first integral is also zero, will be clear when we observe that the integrand is real and is an even function of t when t is real, so that its Laurent expansion in the neighbourhood of zero contains only even powers of t.

Thus the proof of the theorem is complete.

Proof of theorem 2. As in Theorem 1, we observe that

$$\sum\frac{C_n^k(1)}{n^s(n+2\nu)^s}=-\left(\frac{2}{\nu}\right)^{s-\frac{1}{2}}\frac{\Gamma(s+\frac{1}{2})\,\Gamma(1-2s)}{2i\pi}$$

$$\times\int_c\left\{\frac{\cosh t/2}{(\sinh t/2)^k}-e^{-\nu t}\right\}(-t)^{s-\frac{1}{2}}\,I_{s-\frac{1}{2}}(-\nu t)\;dt$$

$$= -\left(\frac{2}{\nu}\right)^{s-\frac{1}{2}} \frac{\Gamma(s+\frac{1}{2})\,\Gamma(1-2s)}{2i\pi}$$

$$\times\left[\int_c \frac{\cosh t/2}{(\sinh t/2)^k}(-t)^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(-\nu t)\,dt - \int_c e^{-\nu t}(-t)^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(-\nu t)\,dt\right] \quad (19)$$

which again shows that the series on the left can be continued analytically with the possible exception of poles at positive integral and half odd integral values of s. The second integral vanishes for these values and also for the negative integral values of s. To see the contribution from the first integral we consider two cases:

CASE (i) when k is even.

$\dfrac{\cosh t/2}{(\sinh t/2)^k}$ is an even function of t when t is real.

$(-t)^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(-\nu t)$ is an even function of t when t is real and s half an odd integer. Therefore the product being even, the first integral also vanishes when s is half an odd integer thus destroying the possible poles at these points. We are thus left with simple poles at $s = 1, 2, \ldots, k/2$.

CASE (ii) when k is odd.

$\dfrac{\cosh t/2}{(\sinh t/2)^k}$ is an odd function of t when t is real.

$(-t)^{s-\frac{1}{2}} I_{s-\frac{1}{2}}(-\nu t)$ is an odd function of t when t is real and s an integer. Therefore the product being even, we observe that the first integral vanishes for these values of s, thus destroying the poles at positive integers and leaving zeros at negative integers.

Therefore the Dirichlet's series represents a meromorphic function with simple poles at $s = k/2,\ k/2-1,\ k/2-2, \ldots$ and trivial zeros at $s = -1, -2, -3, \ldots$

The proof of the theorem is thus complete.

Calculation of residues at the poles and finding the functional equation if any will form the subject of a subsequent note.

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